Precautionary Saving under Different Types of Risks: The Income and Substitution Effect Revisited *

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December 25, 2020

Abstract

We study precautionary saving in a two-period model using Davis's(1989) compensation method, which allows a separation between income and substitution effect when the decision maker faces an increase in risk. We prove that the substitution effect is always negative and therefore, the income effect must then be positive and larger in size to have an increase in saving or a precautionary effect. We then apply the method to different sources of risks like income, interest rate and wealth risk and analyze the magnitude of each effect and find the conditions required to guarantee precautionary saving in each case. We observe that when the utility function is time separable, the conditions to have precautionary effect are the ones we know from the literature. However, our more general setting allows for the possibility of non-linear risk effects. Our results are presented as signs of covariances, which provide a new perspective on the issue of precautionary saving.

*The authors acknowledge the useful comments of Erwin Hansen, Pablo Gutierrez, Jaime Miranda and seminar participants at the University of Chile and Universidad del Desarrollo. The usual disclaimer applies.

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Keywords: Precautionary saving, types of risks, increases in risk, mean-preserving spreads, income effect, substitution effect.
JEL Classification: E21, D81

1 Introduction

When an expected utility decision-maker faces an increase in future risk, he changes his optimal saving decision in order to better accommodate his inter-temporal consumption and risk exposure over time. The usual concept that arises in this type of analysis is the precautionary saving effect, which is the extra saving produced in response to the increase in risk. This effect is guaranteed as long as the third derivative of the utility function is positive (Leland, 1968; Sandmo, 1970; Kimball, 1990) .

Most of the literature on precautionary saving assumes additive separable inter-temporal utility functions and additive risk that in most cases is associated with a shock to future wages. However, when we start thinking about additional real-world cases like, for example, the case of heterogeneous agents (Ponce and Yuki, 2006), the multiplicative risk associated with inter-temporal interest rate risk (Eeckhoudt and Schlesinger, 2008), the case of ambiguous environments in dynamic settings (Gomes, 2014), or even more complex cases like the ones described in Gunning (2010), where developing economies presents non-linear risk shocks due to the credit market constraints and asymmetric information problems, a positive third derivative of the utility function (prudence) may not be enough to cause the precautionary effect (Eeckhoudt and Schlesinger, 2008, Vergara, 2017 and Baiardi et al., 2020).

Our paper is related to Gunning (2010), who studied labor, interest rate and wealth risk and suggested that individuals in developing countries are exposed to great uninsured risks that are mostly non-linear and have different effects on optimal decisions. To study the effect of risk on savings, we use Davis's(1989) compensation method to distinguish between income and substitution effects of a risk increase under different sources of risk. Our paper is also related to Snow (2003) and Machina and Pratt (1997). The former decomposes a mean-preserving spread into a substitution and income effect but only for the case of an additive separable utility function with no applications whatsoever, and the latter extends the Rothschild and Stiglitz (1970, 1971) characterization of a mean-preserving spread to a more general definition with an extension to second-order stochastic dominance. We develop three applications and connect them with recent literature on decision-making under risk through a perspective of using the signs of covariances that appear in optimal saving problems.

2 The Model

We consider a two-period consumption model with a deterministic endowment income w_1 in the first period but an uncertain total income in the second period. The consumer has a utility function $u(c_1, c_2)$, which is assumed to be continuous, increasing and concave in each argument, and at least three-times differentiable. That is, u_1 and u_2 are positive while u_{11} and u_{22} are negative, where the subscripts denote partial derivatives. We assume that $u_{12} = u_{21} > 0$, meaning that consumption levels in the first and second periods interact as complements. Also, $c_1 = w_1 - s$ and $c_2 = g(\theta, w_2, s)$, where θ represents a stochastic shock that affects total income in the second period, w_2 is a non-stochastic exogenous variable (second-period deterministic endowment) and s denotes saving. The function g is increasing and concave in s ($g_s > 0$ and $g_{ss} < 0$) and increasing and concave in θ ($g_\theta > 0$ and $g_{\theta\theta} < 0$). Also, g_s is increasing and concave in θ, that is $g_{s\theta} > 0$ and $g_{s\theta\theta} < 0$.

Let $F(\theta, \tau)$ be the cumulative distribution function of θ, with support in the interval $[a, b]$, where τ is a parameter whose shift represents changes in risk like in Diamond and Stiglitz (1974).

The consumer seeks the optimal savings s^* that maximize the intertemporal utility of consumption, which is achieved by choosing $s^*(\tau)$ such that

$$s^*(\tau) = \operatorname*{Arg\,max}_s \{Eu(c_1, c_2) = \int_a^b u(w_1 - s, g(\theta, w_2, s))dF(\theta, \tau)\} \tag{1}$$

The first and second-order conditions of the consumer's problem are:

$$\int_a^b [-u_1 + u_2 g_s] dF(\theta, \tau) = -Eu_1 + Eu_2 g_s = 0 \tag{2}$$

$$\int_a^b [u_{11} - 2u_{12} g_s + u_{22} g_s^2 + u_2 g_{ss}] dF(\theta, \tau) < 0 \tag{3}$$

For ease of exposition we will use the following definitions:

$$U(s, \theta) = u(w_1 - s, g(\theta, w_2, s)) \tag{4}$$

$$U_s(s, \theta) = -u_1 + u_2 g_s \tag{5}$$

$$U_{s\theta}(s, \theta) = -u_{12} g_\theta + u_{22} g_\theta g_s + u_2 g_{s\theta} \tag{6}$$

$$U_{s\theta\theta}(s, \theta) = -u_{122} g_\theta^2 - u_{12} g_{\theta\theta} + u_{222} g_\theta^2 g_s + u_{22} g_{\theta\theta} g_s + u_{22} g_\theta g_{s\theta} + u_{22} g_\theta g_{s\theta} + u_2 g_{s\theta\theta} \tag{7}$$

2.1 Risk Changes

Following Diamond and Stiglitz (1974) we define an increase in τ as a first-order risk increase if

$$F_\tau(\theta,\tau) > 0 \tag{8}$$

and $F_\tau(a,\tau) = F_\tau(b,\tau) = 0$.

Alternatively, an increase in τ represents a second-order risk increase (mean-preserving spread) if

$$\int_a^b F_\tau(\theta,\tau)d\theta = 0 \tag{9}$$

and

$$T(\theta,\tau) = \int_a^\theta F_\tau(z,\tau)dz \geq 0 \quad \text{for all} \quad a \leq \theta \leq b \tag{10}$$

and also, $T(a,\tau) = T(b,\tau) = 0$. Condition (9) indicates that both distributions have the same mean and condition (10) is the single-crossing property of the mean-preserving spread.

Proposition 1. *(Conditions for Precautionary Effect).*
Let $s^*(\tau)$ be the optimal level of saving that maximizes (1). Then, we have the following two results:

(i) If a shift in τ represents a first-order risk increase, then s^* increases if $U_{s\theta} < 0$.

(ii) If a shift in τ represents a mean-preserving spread, then s^* increases if $U_{s\theta\theta} > 0$

Proof. (i) Using the implicit function theorem in the first-order condition (2) we obtain:

$$\frac{ds^*}{d\tau} = -\frac{\int_a^b U_s dF_\tau(\theta,\tau)}{SOC} \tag{11}$$

where SOC stands for the second-order condition, which is negative as seen in (3). Therefore, $sign(\frac{ds^*}{d\tau}) = sign(\int_a^b U_s dF_\tau(\theta,\tau))$.

Now, integrating expression $\int_a^b U_s dF_\tau(\theta,\tau)$ by parts, we get $-\int_a^b U_{s\theta}F_\tau d\theta$, which is positive as long as $U_{s\theta} < 0$, which guarantees the precautionary effect for a first-order risk increase and completes part (i) of the proof.

(ii) Again, integrating expression $-\int_a^b U_{s\theta}F_\tau d\theta$ by parts, we get $\int_a^b U_{s\theta\theta}T(\theta,\tau)d\theta$, which is positive as long as $U_{s\theta\theta} > 0$, which guarantees the precautionary effect for the mean-preserving spread and completes part ii) of the proof. □

3 A Specific Definition of a Mean-Preserving Spread

Let us write the random variable θ as:

$$\theta = \mu_\theta + \gamma\epsilon \tag{12}$$

where μ_θ is the mean of θ, γ is a positive scalar that represents a spread parameter and ϵ is a zero-mean random variable. Note that an increase in γ generates a mean-preserving spread in θ. In this case the consumer's problem becomes

$$s^*(\gamma) = \operatorname*{Arg\,max}_s \{Eu(w_1 - s, g(\mu_\theta + \gamma\epsilon, w_2, s)\} \tag{13}$$

Proposition 2. *Let $s^*(\gamma)$ be the optimal level of saving that maximizes (13). A mean-preserving spread (increase in γ) induces an increase (decrease) in s^* as long as $cov(U_{\theta s}, \epsilon) > (<) 0$.*

Proof. By implicitly differentiating the first order condition derived from problem (13) we obtain:

$$\frac{ds^*}{d\gamma} = -\frac{-Eu_{12}g_\theta\epsilon + Eu_{22}g_s g_\theta\epsilon + Eu_2 g_{s\theta}\epsilon}{SOC} = -\frac{cov(U_{\theta s}, \epsilon)}{SOC} \tag{14}$$

which proves that, as long as $cov(U_{\theta s}, \epsilon) > 0$, we have a precautionary effect. □

Note that $cov(U_{\theta s}, \epsilon) = cov((-u_{12} + u_{22}g_s)g_\theta, \epsilon) + cov(u_2 g_{\theta s}, \epsilon)$ below we will show that these two covariance can be linked to the income and substitution effects respectively by using the Davis(1989)'s compensation method.

3.1 The Compensation Method

The compensation method (in a risky context) asks for the amount (given to or taken from w_2) sufficing to obtain the same amount of c_2^* if the choice of s remains s^*. This exercise keeps the initial expected utility constant and it provides the substitution effect under an increase in risk.

When the consumer experiences a mean-preserving spread, precautionary saving will induce an increase in saving. This increase in saving will, in turn, induce an increase in expected c_2^*. Therefore, the compensation method will provide a negative substitution effect (to go back to initial c_2^*). Consequently, the income effect must be positive and larger in size than the substitution effect to have precautionary saving.

Let us assume that $w_2 = \varphi(\gamma)$. The consumption in the second period is represented by

$$c_2 = g(\mu_\theta + \gamma\epsilon, \varphi(\gamma), s^*) \tag{15}$$

The compensation method implies the following random rule

$$\frac{\partial c_2}{\partial \gamma} = g_\theta \epsilon + g_{w_2}\varphi_\gamma = 0 \tag{16}$$

Solving for φ_γ we get:

$$\varphi_\gamma = -\frac{g_\theta \epsilon}{g_{w_2}} \tag{17}$$

In order to obtain the substitution effect we plug $\varphi(\gamma)$ into the first order condition (2) and apply the implicit function theorem to get

$$\frac{ds^*}{d\gamma}|_{comp} = -\frac{(-Eu_{12} + Eu_{22}g_s)(g_\theta \epsilon + g_{w_2}\varphi_\gamma) + Eu_2(g_{s\theta}\epsilon + g_{sw_2}\varphi_\gamma)}{SOC} \tag{18}$$

Considering the compensation rule (16) and following Davis's(1989) assumption that $g_{sw_2} = 0$, equation (18) is reduced to

$$\frac{ds^*}{d\gamma}|_{comp} = -\frac{Eu_2 g_{s\theta}\epsilon}{SOC} = -\frac{cov(u_2 g_{s\theta}, \epsilon)}{SOC} \tag{19}$$

and consequently, the substitution effect is linked to the sign of $cov(u_2 g_{s\theta}, \epsilon)$.

Please recall that equation (14) provides the total effect on s^* under a mean-preserving spread. In particular a positive sign of $cov(U_{\theta s}, \epsilon) = cov((-u_{12}+u_{22}g_s)g_\theta, \epsilon)+cov(u_2 g_{\theta s}, \epsilon)$ guarantees precautionary savings. Also, the compensation method provides the substitution effect by $cov(u_2 g_{\theta s}, \epsilon)$. Therefore, $cov((-u_{12} + u_{22}g_s)g_\theta, \epsilon)$ is the income effect. The sign of each covariance component is obtained by differentiating the following expressions:

$$\frac{\partial(-u_{12} + u_{22}g_s)g_\theta}{\partial \epsilon} = \{-u_{122}g_\theta^2 - u_{12}g_{\theta\theta} + u_{222}g_\theta^2 g_s + u_{22}g_{\theta\theta}g_s + u_{22}g_\theta g_{\theta s}\}\gamma \tag{20}$$

$$\frac{\partial u_2 g_{s\theta}}{\partial \epsilon} = \{u_{22}g_\theta g_{s\theta} + u_2 g_{s\theta\theta}\}\gamma \tag{21}$$

Note that the substitution effect is actually negative, which is given by the sign of (21). Also, by adding (20) and (21) and dividing by γ we get

$$U_{s\theta\theta} = \underbrace{\{-u_{122}g_\theta^2 - u_{12}g_{\theta\theta} + u_{222}g_\theta^2 g_s + u_{22}g_{\theta\theta}g_s + u_{22}g_\theta g_{s\theta}}_{\text{Income effect}} + \underbrace{u_{22}g_\theta g_{s\theta} + u_2 g_{s\theta\theta}}_{\text{Substitution effect}} \tag{22}$$

and we already know from previous propositions that as long as $U_{s\theta\theta} > 0$, precautionary saving is guaranteed under a mean-preserving spread.

Table 1 summarizes the income and substitution effects for different types of utility functions and risks.

Table 1: Precautionary effect

Cases	Income Effect	Substitution Effect
Additively separable utility	$u_{222}g_\theta^2 g_s + u_{22}g_{\theta\theta}g_s + u_{22}g_\theta g_{\theta s}$	$u_{22}g_\theta g_{\theta s} + u_2 g_{s\theta\theta}$
Linear risk	$-u_{122}g_\theta^2 + u_{222}g_\theta^2 g_s + u_{22}g_\theta g_{\theta s}$	$u_{22}g_\theta g_{\theta s}$
Additively separable utility and linear risk	$u_{222}g_\theta^2 g_s + u_{22}g_\theta g_{\theta s}$	$u_{22}g_\theta g_{\theta s}$
Additive risk	$-u_{122}g_\theta^2 + u_{222}g_\theta^2 g_s$	0
Additively separable utility and additive risk	$u_{222}g_\theta^2 g_s$	0

At this point, it is important to mention that $g_{\theta\theta}$ and $g_{s\theta\theta}$ allow the possibility of having non-linear risks in our analysis and, as we mentioned before, non-linearities are often present in developing economies where great risks are uninsured (Gunning, 2010) and increases in risk have been proved to decrease saving instead of inducing a precautionary effect (Dercon, 2005).

Also, non-linear risk allows the theorization of new risks that are not commonly treated in the economic literature, like the risk of Covid 19 and its devastating economic consequences worldwide. In the end, equation (22) is a general expression that captures not only traditional risk analysis commonly observed in the literature, but also new risks with more complex functional forms as well.

4 Applications

In this section we will study three sources of risk. The classic labor income risk with a linear and additive shock. The case of interest-rate risk with a multiplicative and linear shock like in Eeckhoudt and Schlesinger (2008); and finally, we will study a wealth risk shock as in Gunning (2010).

4.1 Labor income risk

Let us assume that the second-period consumption is represented by

$$c_2 = g(\theta, w_2, s) = w_2\theta + sR \tag{23}$$

where $\theta = \mu_\theta + \gamma\epsilon$ and R is the non-random gross interest rate. In this case, $U(s,\theta) = u(w_1 - s, w_2\theta + sR)$ and since in this case $g_\theta = w_2$, $g_s = R$ and $g_{\theta s} = g_{\theta\theta s} = 0$, then $cov(U_{s\theta}, \epsilon) = cov((-u_{12} + u_{22}R)w_2, \epsilon)$. Therefore, in this case the precautionary effect is determined only by the income effect. Note that $cov((-u_{12} + u_{22}R)w_2, \epsilon) > 0$ if and only if $\dfrac{\partial U_{s\theta}}{\partial \epsilon} = (-u_{122} + u_{222}R)w_2^2\gamma > 0$. If $u(c_1, c_2)$ is additively separable ($u_{122} = 0$), then precautionary saving takes place whenever the decision-maker is prudent, i.e. $u_{222} > 0$.

4.2 Interest-rate risk

Let's assume that the consumption in the second-period is represented by the following equation:

$$c_2 = g(\theta, w_2, s) = w_2 + s\theta R \tag{24}$$

where $g_s = \theta R$, $g_\theta = sR$, $g_{s\theta} = R$ and $g_{s\theta\theta} = 0$. In this case $U(\theta, s) = u(w_1 - s, w_2 + s\theta R)$ and $cov(U_{\theta s}, \epsilon) = cov((-u_{12} + u_{22}\theta R)sR, \epsilon) + cov(u_2 R, \epsilon)$. The precautionary effect is determined by both: the income effect and the substitution effect. The substitution effect is negative since the sign of the $cov(u_2 R, \epsilon)$ can be obtained from the sign of

$$\frac{\partial u_2 R}{\partial \epsilon} = u_{22} s R^2 \gamma \tag{25}$$

Therefore, to have a precautionary effect, the income effect must be positive and greater in magnitude than the substitution effect. This means that

$$-u_{122}(sR)^2\gamma + u_{222}(sR)^2\gamma\theta R + u_{22} s R^2 \gamma > -u_{22} s R^2 \gamma \tag{26}$$

By working on inequality (26) and assuming a separable utility function, we obtain the following condition to guarantee the precautionary effect

$$\frac{-u_{222}}{u_{22}} s\theta R = \frac{-u_{222}}{u_{22}}[C_2 - w_2] > 2 \tag{27}$$

where $\frac{-u_{222}}{u_{22}}$ is the absolute prudence coefficient.

4.3 Wealth risk

Following Gunning (2010) we define second-period consumption by

$$c_2 = g(\theta, w_2, s) = \theta(w_2 + (1-\delta)s + h(s)) = \theta W \tag{28}$$

where W stands for total wealth, $(1-\delta)s$ the expected value of assets, and $h(s)$ is the expected value of capital income. The function $h(s)$ is increasing ($h' > 0$) and concave ($h'' < 0$), with $h(0) = 0$. Also, $g_s = \theta W_s$, $g_\theta = W$ and $g_{s\theta} = W_s$, where $W_s = (1-\delta) + h'$.

In this case, $U(\theta, s) = u(w - s, \theta W))$ and $cov(U_{s\theta}, \epsilon) = cov((-u_{12} + u_{22}\theta W_s)W, \epsilon) + cov(u_2 W_s, \epsilon)$. We know that we have a precautionary effect as long as $cov(U_{s\theta}, \epsilon) > 0$. In this case the substitution effect is negative because

$$\frac{\partial u_2 W_s}{\partial \epsilon} = u_{22} W_s \gamma W < 0 \tag{29}$$

and just like in the previous case, to have a precautionary effect, the income effect must be positive and greater in magnitude than the substitution effect. This means that in this case we need

$$[-u_{122}W\gamma + u_{222}\theta W_s W\gamma + u_{22}W_s\gamma]W > -u_{22}W_s\gamma W \tag{30}$$

By working on inequality (30) and assuming separable utility function, we obtain the following condition to guarantee the precautionary effect

$$\frac{-u_{222}}{u_{22}}\theta W = \frac{-u_{222}}{u_{22}}c_2 > 2 \tag{31}$$

where $\dfrac{-u_{222}}{u_{22}}$ is the relative prudence coefficient.

Table 2 summarizes the income effect and the substitution effect for the three types of risks described above.

Table 2: Income and substitution effect for different sources of risks

Sources	$g(\theta, w_2, s)$	Income effect	Substitution effect
Labor income risk	$w_2\theta + sR$	$(-u_{122} + u_{222}R)w_2^2\gamma$	0
Interest-rate risk	$w_2 + s\theta R$	$(s^2(-u_{122} + u_{222}\theta R) + u_{22}s)R^2\gamma$	$u_{22}sR^2\gamma$
Wealth risk	$\theta(w_2 + (1-\delta)s + h(s))$	$[-u_{122}W\gamma + u_{222}\theta W_s W\gamma + u_{22}W_s\gamma]W$	$u_{22}W_s W\gamma$

5 Conclusion

Using Davis's(89) compensation method we have seen that when the consumer faces a mean-preserving spread, precautionary saving can be divided into an income effect and a substitution effect, and these effects can be represented by covariances. The substitution effect is negative, so the income effect must then be positive and larger in size than the substitution effect to have increases in savings and, therefore, the precautionary effect. Our analysis allows for the treatment of more complex risk structures since non-linear risks are a possibility in our general formulation of the effects of risk increases on savings.

We have studied the effects of different sources of risk like income, interest rate and wealth. We then define the conditions for each case to have precautionary savings.

Finally, when we assume additive separability utility functions, the conditions for precautionary savings relate to the concept of prudence and relative prudence, which is consistent with the current literature on savings under risk.

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